ISRAEL JOURNAL OF MATHEMATICS **160** (2007), 1–21
DOI: 10.1007/s11856-007-0053-3

QUASI-LIPSCHITZ EQUIVALENCE OF FRACTALS

BY

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ABSTRACT

The paper proves that if E and F are dust-like C^1 self-conformal sets with $0 < \mathcal{H}^{\dim_H E}(E), \mathcal{H}^{\dim_H F}(F) < \infty$, then there exists a bijection $f\colon E \to F$ such that

$$\frac{(\dim_H F) \log |f(x) - f(y)|}{(\dim_H E) \log |x - y|} \to 1$$

uniformly as $|x-y| \to 0$. It is also proved that a self-similar arc is Hölder equivalent to $[0,1]$ if and only if it is a quasi-arc.

1. Introduction

A non-empty compact set E is said to be the invariant set of a family of bijective contractions $\{f_i\}_{i=1}^m$, if $E = \bigcup_{i=1}^m f_i(E)$ ([7]). In particular, when the union $\bigcup_{i=1}^m f_i(E)$ is a disjoint union, we call E a **dust-like** set ([6]). The dust-like invariant sets are totally disconnected.

For connected case, we consider **arcs**, the homeomorphic images of $[0,1]$. For some contracting similitudes $\{S_i\}_{i=1}^m$, we call $\gamma = \bigcup_{i=1}^m S_i(\gamma)$ a self-similar arc, if γ is an arc and $S_i(\gamma) \cap S_j(\gamma) = \varnothing$ if $|i-j| > 1$, $S_i(\gamma) \cap S_j(\gamma)$ is a singleton if $|i-j| = 1$ ([9]). An arc γ is called a quasi-arc if the diameter $diam(\gamma(x,y)) \leq C|x-y|$ for all $x, y \in \gamma$, where C is a constant and $\gamma(x,y)$ is the subarc lying between x and y.

* Research supported by National Natural Science Foundation of China (No. 10241003, 10301029) and Morningside Center of Mathematics in Beijing.
Received March 6, 2005 and in revised form January 16, 2006

Definition 1: Suppose $V \subset \mathbb{R}^n$ is open. A C^1-mapping $g\colon V \to \mathbb{R}^n$ is said to be conformal if the differential $Dg(x)\colon \mathbb{R}^n \to \mathbb{R}^n$ is a similarity transformation for each $x \in V$. Furthermore, we say g is of $C^{1+\gamma}$ conformal class, if $Dg(x)$ also satisfies the Hölder condition with exponent $\gamma > 0$, i.e., $\|Dg(x) - Dg(y)\| \leq c|x-y|^\gamma$ for all $x, y \in V$, where $c > 0$ is a constant.

A compact set $F \subset \mathbb{R}^n$ is called a ($C^{1+\gamma}$) self-conformal set if F is the invariant set of ($C^{1+\gamma}$) conformal and bijective contractions $\{f_i\}_{i=1}^m$ defined on an open neighborhood V of F.

There are two elementary topological objects homeomorphic to the self-similar arcs and dust-like self-conformal sets respectively: the unit interval $[0,1]$ in $\mathbb{R}^1$ and the symbolic system

$$C = \{\{x_i\}_i = x_1 x_2 \cdots | x_i = 0 \text{ or } 1\},$$

and where C is equipped with a metric d satisfying $d(\{x_i\}_i, \{y_i\}_i) = 2^{-j}$ with $j = \min\{i : x_i \neq y_i\}$ for distinct elements $\{x_i\}_i, \{y_i\}_i \in C$. Then $\dim_H C = 1$. Here $[0,1]$ is path connected and C is totally disconnected.

In this paper, we will use the above two objects to represent self-similar arcs and dust-like self-conformal sets, respectively.

The concept of the Lipschitz equivalence is important in the research of fractals ([1], [2], [3] and [4]). Two compact sets $E \subset \mathbb{R}^{n_1}, F \subset \mathbb{R}^{n_2}$ are **Lipschitz equivalent** if there is a bijection $f\colon E \to F$ such that

$$c \cdot |x-y| \leq |f(x) - f(y)| \leq c^{-1} \cdot |x-y| \quad \text{for all } x, y \in E, \tag{1.1}$$

where $c > 0$ is a constant, and $|z_1 - z_2|$ is the Euclidean distance between points z_1 and z_2.

In [6], Falconer and Marsh introduced a **weaker** equivalence named nearly Lipschitz equivalence. E and F are said to be **nearly Lipschitz equivalent** if for each $\eta \in (0,1)$, there is a bijection $f_\eta\colon E \to F$ such that

$$c_\eta \cdot |x-y|^{1/\eta} \leq |f_\eta(x) - f_\eta(y)| \leq c_\eta^{-1} \cdot |x-y|^\eta \quad \text{for all } x, y \in E, \tag{1.2}$$

where the constant c_η is dependent on η, E and F. In the category of compact sets, Lipschitz equivalence implies nearly Lipschitz equivalence.

It is well-known that if E and F are nearly Lipschitz equivalent, then $\dim_H E = \dim_H F$. There are some related results:

(1) Suppose E, F are dust-like C^1 self-conformal sets in Euclidean spaces. Then $\dim_H E = \dim_H F$ if and only if E and F are nearly Lipschitz equivalent ([6], [10]).

(2) Two quasi-self-similar circles have the same Hausdorff dimension if and only if they are Lipschitz equivalent ([5]).

(3) There are two self-similar arcs with the same Hausdorff dimension, though they are not nearly Lipschitz equivalent ([9]).

In addition, the invariant property of the (nearly) Lipschitz equivalence is useful, for example, uniform perfectness ([8], [11], [12]), nearly uniform perfectness ([10]) and porosity ([13]) of compact sets in metric spaces.

A new concept named quasi-Lipschitz equivalence, which is weaker than Lipschitz equivalence and stronger than nearly Lipschitz equivalence, is defined as follows.

Definition 2: Two compact sets E and F of Euclidean spaces are **quasi-Lipschitz equivalent** if there is a bijection $f\colon E \to F$ such that for every $\varepsilon > 0$, there exists $\delta > 0$ satisfying

$$\left|\frac{\log|f(x)-f(y)|}{\log|x-y|} - 1\right| < \varepsilon \tag{1.3}$$

whenever $x, y \in E$ with $0 < |x-y| < \delta$.

By a simple calculation, it is easy to see that Definition 2 has the following equivalent form.

Definition 3: Two compact sets E and F of Euclidean spaces are **quasi-Lipschitz equivalent** if there is a bijection $f\colon E \to F$ such that for any $\eta \in (0,1)$,

$$c_\eta \cdot |x-y|^{1/\eta} \le |f(x)-f(y)| \le c_\eta^{-1} \cdot |x-y|^\eta \quad \text{for all } x, y \in E, \tag{1.4}$$

where the constant c_η is dependent on η, E and F.

Compared with (1.2), (1.4) shows that quasi-Lipschitz equivalence is stronger than nearly Lipschitz equivalence.

The main **representation theorems** of this paper are stated as follows. Firstly, we will deal with the totally disconnected case.

THEOREM 1: *Suppose E is a dust-like self-conformal set satisfying*

$$0 < \mathcal{H}^{\dim_H E}(E) < \infty.$$

Then there is a bijection $f\colon E \to C$ such that for every $\varepsilon > 0$, there exists $\delta > 0$ satisfying

$$\left|\frac{\log d(f(x), f(y))}{\dim_H E \cdot \log|x-y|} - 1\right| < \varepsilon$$

whenever $x, y \in E$ *with* $0 < |x-y| < \delta$.

As a consequence, we have:

THEOREM 2: *Suppose* E, F *are dust-like self-conformal sets in Euclidean spaces with* $0 < \mathcal{H}^{\dim_H E}(E), \mathcal{H}^{\dim_H F}(F) < \infty$. *Then there is a bijection* $f\colon E \to F$ *such that for every* $\varepsilon > 0$, *there exists* $\delta > 0$ *satisfying*

$$\Big|\frac{\log|f(x)-f(y)|}{\log|x-y|} - \frac{\dim_H E}{\dim_H F}\Big| < \varepsilon \tag{1.5}$$

whenever $x, y \in E$ *with* $0 < |x-y| < \delta$.

In particular, $\dim_H E = \dim_H F$ *if and only if* E *and* F *are quasi-Lipschitz equivalent.*

Remark 1: If $f\colon E \to F$ is a bijection satisfying $\frac{\log|f(x)-f(y)|}{\log|x-y|} \to t$ uniformly as $|x-y| \to 0$, then $t = \dim_H E/\dim_H F$ by the definition of the Hausdorff dimension.

For any dust-like $C^{1+\gamma} (\gamma > 0)$ self-conformal set E, we always have $0 < \mathcal{H}^{\dim_H E}(E) < \infty$. Therefore, we have the following corollary.

COROLLARY 1: *Suppose* E, F *are dust-like* $C^{1+\gamma}$ *self-conformal sets with* $\gamma > 0$. *Then there is a bijection* $f\colon E \to F$ *such that for every* $\varepsilon > 0$, *there exists* $\delta > 0$ *satisfying*

$$\Big|\frac{\log|f(x)-f(y)|}{\log|x-y|} - \frac{\dim_H E}{\dim_H F}\Big| < \varepsilon \tag{1.6}$$

whenever $x, y \in E$ *with* $0 < |x-y| < \delta$.

In particular, $\dim_H E = \dim_H F$ *if and only if* E *and* F *are quasi-Lipschitz equivalent.*

Remark 2: In particular, when $\dim_H E = \dim_H F$, as an equivalent form, Corollary 1 shows for all $x, y \in E$ and any $\eta \in (0,1)$ that

$$c_\eta \cdot |x-y|^{1/\eta} \le |f(x)-f(y)| \le c_\eta^{-1} \cdot |x-y|^\eta,$$

where c_η is dependent on η. Inequality (1.2) is proved in [6]. The difference between nearly Lipschitz equivalence and quasi-Lipschitz equivalence is that the family $\{f_\eta\}_{\eta\in(0,1)}$ of bijections in (1.2) is replaced by one bijection f in (1.4). This is the difficulty of the proof.

Remark 3: In Theorems 1 and 2, we only use the C^1 conformal condition, but we do not need the $C^{1+\gamma}$ conformal condition with $\gamma > 0$. If h is a conformal mapping in $\mathbb{R}^n$ with $n \geq 2$, then h is analytic, i.e., $h \in C^\infty$. That means the C^1 conformal condition is meaningful only in dimension one. In addition, the counterexample in [10] shows that Theorems 1 and 2 do not hold for the invariant sets of bi-Lipschitz contraction.

Notice that in [9] a special self-similar arc is constructed such that it fails to be a quasi-arc, and a sufficient condition for a self-similar arc to be a quasi-arc is obtained.

Definition 4: Suppose $E \subset \mathbb{R}^{n_1}$ and $F \subset \mathbb{R}^{n_2}$ are compact. We say that E is Hölder equivalent to F, if there are constants $\alpha, c > 0$ and a bijection $f\colon E \to F$ such that for all $x, y \in E$,

$$c \cdot |x-y| \leq |f(x) - f(y)|^\alpha \leq c^{-1} \cdot |x-y|.$$

For the path connected case, we have the following result.

THEOREM 3: *A self-similar arc is Hölder equivalent to the unit interval* $[0,1]$ *if and only if it is a quasi-arc.*

We organize the paper as follows. In Section 2, it is pointed out that Theorem 2 is a consequence of Theorem 1, and some uniform estimations (Lemmas 1–4) for C^1 IFS are obtained. Section 3 is the proof of Theorem 1. Within Section 4, the proof of Theorem 3 is provided.

2. Preliminaries

Let $C = \{\{x_i\}_i = x_1 x_2 \cdots \mid x_i = 0 \text{ or } 1\}$ be the canonical Cantor set equipped with a metric d satisfying $d(\{x_i\}_i, \{y_i\}_i) = 2^{-j}$ where $j = \min\{i : x_i \neq y_i\}$ for distinct elements $\{x_i\}_i, \{y_i\}_i \in C$. Then $\dim_H C = 1$.

As in [6], for Theorem 2, we need only to prove Theorem 1:

Suppose E is a dust-like self-conformal set with $0 < H^{\dim_H E}(E) < \infty$. *Then there is a bijection* $f\colon E \to C$ *such that for every* $\varepsilon > 0$, *there exists* $\delta > 0$ *satisfying*

$$\left| \frac{\log d(f(x), f(y))}{\dim_H E \cdot \log |x-y|} - 1 \right| < \varepsilon \tag{2.1}$$

whenever $x, y \in E$ *with* $0 < |x-y| < \delta$.

Theorem 1 $\Longrightarrow$ Theorem 2: In fact, suppose Theorem 1 is proved; then for two dust-like conformal sets E and F, there are two bijections $f\colon E \to C$ and $g\colon C \to F$ satisfying for all $x_1 \neq x_2 \in E$ and $y_1 \neq y_2 \in C$,

$$\frac{\log d(f(x_1), f(x_2))}{\dim_H E \cdot \log |x_1 - x_2|} \to 1 \text{ uniformly} \quad \text{as } |x_1 - x_2| \to 0,$$
$$\frac{\dim_H F \cdot \log |g(y_1) - g(y_2)|}{\log d(y_1, y_2)} \to 1 \text{ uniformly} \quad \text{as } d(y_1, y_2) \to 0.$$

Then $h = g \circ f\colon E \to F$ is a bijection satisfying for $x_1 \neq x_2 \in E$,

$$\frac{\dim_H F \cdot \log |h(x_1) - h(x_2)|}{\dim_H E \cdot \log |x_1 - x_2|} \to 1 \text{ uniformly} \quad \text{as } |x_1 - x_2| \to 0.$$

And thus Theorem 2 is proved. ∎

We also need to establish some results about C^1 self-conformal sets.

Remark 4: The dust-like self-conformal sets are complicated in general, though cookie-cutter sets in $\mathbb{R}$ are relatively simple (e.g., see [4]). In fact, for any cookie-cutter set in $\mathbb{R}$, there are many gaps, basic intervals pairwise disjoint, with simple estimation of their lengths.

Suppose $E \subset \mathbb{R}^n$ is the dust-like invariant set of a family $\{\varphi_i\}_{i=1}^m$ of C^1 conformal and bijective contractions with $0 < \mathcal{H}^s(E) < \infty$, where $s = \dim_H E > 0$. For every i the mapping φ_i is defined on some open set U such that $D\varphi_i$ is continuous in $\bar{V}$ for some bounded open set V satisfying $E \subset V \subset \bar{V} \subset U$, where the compact subset $\bar{V}$ is the closure of V.

Let $\Sigma^* = \{i_1 \cdots i_k : k \in \mathbb{N}, 1 \leq i_r \leq m \text{ for all } r\}$ be the set of all the finite sequences, and Σ_m the set of all the infinite sequences. A subset $\mathfrak{F}$ of Σ^* is called a **cut-set** ([6]) if, for every infinite sequence $(i_1 i_2 \cdots) \in \Sigma_m$, there exists a unique integer k such that $(i_1 i_2 \cdots i_k) \in \mathfrak{F}$. For any $i_1 i_2 \cdots i_k \in \Sigma^*$, let $[i_1 i_2 \cdots i_k] = \{j_1 j_2 \cdots \in \Sigma_m : j_r = i_r \text{ for } 1 \leq r \leq k\}$, which is called a **cylinder** in Σ_m.

Similarly, for the canonical Cantor set C, we also obtain the set Π^* of all the finite sequences composed of 0 and 1; then we also have the concepts of 'cut-set' and 'cylinder' naturally.

For every sequence $i_1 \cdots i_n \in \Sigma^*$, write $\varphi_{i_1 \cdots i_n} = \varphi_{i_1} \circ \cdots \circ \varphi_{i_n}$ and $E_{i_1 \cdots i_n} = \varphi_{i_1 \cdots i_n}(E)$. If $\{x\} = \bigcap_{n \geq 1} \varphi_{i_1 \cdots i_n}(E)$ for an infinite sequence $i_1 \cdots i_n \cdots$, we denote $x = \varphi_{i_1 \cdots i_n \cdots}(E)$.

LEMMA 1: *There is a decreasing sequence* $\{\delta_k\}_k$ *with* $\lim_{k\to\infty}\delta_k = 0$ *such that for any* $n \geq k$,

$$\Big|\frac{\log \|D_{w_1}\varphi_{i_1\cdots i_n}\|}{\log \|D_{w_2}\varphi_{i_1\cdots i_n}\|} - 1\Big| \leq \delta_k,$$

$$\Big|\frac{\log \mathcal{H}^s(E_{i_1\cdots i_n})}{s\log \|D_{w_2}\varphi_{i_1\cdots i_n}\|} - 1\Big| \leq \delta_k,$$

$$\Big|\frac{\log \mathcal{H}^s(E_{i_1\cdots i_n})}{\log \mathcal{H}^s(E_{i_1\cdots i_{n-1}})} - 1\Big| \leq \delta_k,$$

$$\Big|\frac{\log diam(E_{i_1\cdots i_n})}{\log \|D_{w_2}\varphi_{i_1\cdots i_n}\|} - 1\Big| \leq \delta_k.$$

whenever $i_1\cdots i_n \in \Sigma^*$ *and* $w_1, w_2 \in \bar{V}$.

Proof: **(1)** Since $\{\varphi_i\}_i$ are bijective and conformal contractions, we may assume that

$$0 < \rho' \leq \frac{|\varphi_i(x_1) - \varphi_i(x_2)|}{|x_1 - x_2|} \leq \rho < 1 \quad \text{for all distinct } x_1, x_2 \in U. \tag{2.2}$$

There exists $\delta > 0$ small enough such that

$$E + \delta = \{x : d(x, E) \leq \delta\} \subset V.$$

Then we can select an integer k_0 such that $\rho^{k_0} diam(\bar{V}) \leq \delta$. And thus for all $k \geq k_0$,

$$diam[\varphi_{i_1\cdots i_k}(\bar{V})] \leq \rho^k diam(\bar{V}) \leq \rho^{k_0} diam(\bar{V}) \leq \delta, \tag{2.3}$$

which implies that for $x, y \in \varphi_{i_1\cdots i_k}(\bar{V})$ $(\supset \varphi_{i_1\cdots i_k}(E))$, the segment $[x, y]$ between x and y is contained in $E + \delta$.

Since $\{\varphi_i\}_i$ are contractions, for any sequence $i_1\cdots i_k$,

$$\varphi_{i_1\cdots i_k}(E + \delta) \subset E + \rho^k\delta \subset E + \delta. \tag{2.4}$$

It follows from the continuity of $\{D\varphi_i\}_i$ and the following estimation,

$$0 < \min_{1\leq j\leq m}\min_{z\in\bar{V}} \|D_z\varphi_j\| \leq \max_{1\leq j\leq m}\max_{z\in\bar{V}} \|D_z\varphi_j\| < 1, \tag{2.5}$$

that

$$\frac{\log \|D_x\varphi_i\|}{\log \|D_y\varphi_i\|} \to 1 \text{ uniformly} \quad \text{as } |x - y| \longrightarrow 0. \tag{2.6}$$

Now, since

$$diam(\varphi_{i_r\cdots i_1}(\bar{V})) \leq \rho^r diam(\bar{V}) \to 0 \quad \text{as } r \to \infty, \tag{2.7}$$

by using (2.5), (2.6) and the chain rule, we have

$$\frac{\log\|D_{x_1}\varphi_{i_k\cdots i_1}\|}{\log\|D_{x_2}\varphi_{i_k\cdots i_1}\|}=\frac{\sum_{r=1}^{k}\log\|D_{[\varphi_{i_{r-1}\cdots i_1}(x_1)]}\varphi_{i_r}\|}{\sum_{r=1}^{k}\log\|D_{[\varphi_{i_{r-1}\cdots i_1}(x_2)]}\varphi_{i_r}\|}\longrightarrow 1 \text{ uniformly} \quad \text{as } k\to\infty.$$

That means

$$\frac{\log\|D_{w_1}\varphi_{i_1\cdots i_n}\|}{\log\|D_{w_2}\varphi_{i_1\cdots i_n}\|}\to 1 \text{ uniformly} \quad \text{as } n\to\infty. \tag{2.8}$$

Consequently, using the chain rule, we have

$$\frac{\log\|D_{w_1}\varphi_{i_1\cdots i_n(j_1\cdots j_{k_0})}\|}{\log\|D_{w_2}\varphi_{i_1\cdots i_n}\|}\to 1 \text{ uniformly} \quad \text{as } n\to\infty. \tag{2.9}$$

(2) For $x_1,x_2\in E$,

$$|\varphi_{i_1\cdots i_k(j_1\cdots j_{k_0})}(x_1)-\varphi_{i_1\cdots i_k(j_1\cdots j_{k_0})}(x_2)| = \|D_\zeta\varphi_{i_1\cdots i_k}\|\cdot|\varphi_{j_1\cdots j_{k_0}}(x_1)-\varphi_{j_1\cdots j_{k_0}}(x_2)|$$

where $\zeta\in V$ lies in the segment $[\varphi_{j_1\cdots j_{k_0}}(x_1),\varphi_{j_1\cdots j_{k_0}}(x_2)]\subset E+\delta\subset V$. Notice that by (2.2),

$$(\rho')^{k_0}|x_1-x_2|\le|\varphi_{j_1\cdots j_{k_0}}(x_1)-\varphi_{j_1\cdots j_{k_0}}(x_2)|\le\rho^{k_0}|x_1-x_2|.$$

Therefore, for $\mathbf{i}=i_1\cdots i_k$ and $x_1,x_2\in E$,

$$\|D_\zeta\varphi_{\mathbf{i}}\|(\rho')^{k_0}\le\frac{|\varphi_{\mathbf{i}(j_1\cdots j_{k_0})}(x_1)-\varphi_{\mathbf{i}(j_1\cdots j_{k_0})}(x_2)|}{|x_1-x_2|}\le\|D_\zeta\varphi_{\mathbf{i}}\|(\rho)^{k_0}. \tag{2.10}$$

Since $E_{i_1\cdots i_k(j_1\cdots j_{k_0})}=\varphi_{i_1\cdots i_k(j_1\cdots j_{k_0})}(E)$, by the above estimation for the Lipschitz mapping $\varphi_{i_1\cdots i_k(j_1\cdots j_{k_0})}$, we have

$$\begin{aligned}\min_{x\in\bar V}\|D_x\varphi_{\mathbf{i}}\|^s(\rho')^{sk_0}\mathcal{H}^s(E)&\le\mathcal{H}^s(E_{\mathbf{i}(j_1\cdots j_{k_0})})\\&\le\max_{x\in\bar V}\|D_x\varphi_{\mathbf{i}}\|^s(\rho)^{sk_0}\mathcal{H}^s(E).\end{aligned}$$

Therefore it follows from (2.8) and (2.9) that

$$\frac{\log\mathcal{H}^s(E_{i_1\cdots i_n})}{s\log\|D_{w_2}\varphi_{i_1\cdots i_n}\|}\to 1 \text{ uniformly} \quad \text{as } n\to\infty. \tag{2.11}$$

(3) By using (2.11), to show

$$\frac{\log\mathcal{H}^s(E_{i_1\cdots i_k})}{\log\mathcal{H}^s(E_{i_1\cdots i_{k-1}})}\to 1, \tag{2.12}$$

we need only verify the following formula:

$$\frac{\log \|D_w \varphi_{i_1\cdots i_k}\|}{\log \|D_{[\varphi_{i_k}(w)]}\varphi_{i_1\cdots i_{k-1}}\|} \to 1 \quad \text{for } w \in E,$$

which follows from the chain rule for differentiation.

(4) By (2.10), for $\mathbf{i} = i_1 \cdots i_k$,

$$\min_{x\in\bar{V}} \|D_x\varphi_{\mathbf{i}}\|(\rho')^{k_0} diam(E) \le diam(E_{\mathbf{i}(j_1\cdots j_{k_0})}) \le \max_{x\in\bar{V}} \|D_x\varphi_{\mathbf{i}}\|(\rho)^{k_0} diam(E).$$

It follows from (2.8) and (2.9) that

$$\frac{\log[diam(E_{i_1\cdots i_n})]}{\log \|D_{w_2}\varphi_{i_1\cdots i_n}\|} \to 1 \text{ uniformly.} \quad \blacksquare$$

Lemma 2: *Suppose $\{a_n\}_n$ is a sequence with $\lim_{n\to\infty} a_n = 0$, $a_n > 0$ and $a_{n+1} \le a_n$ for all n. Let $c > 0$. Then there is an increasing sequence $b_n \uparrow \infty$ such that as $n \to \infty$,*

$$n/b_n \to 0, \quad b_{n+1}/b_n \to 1, \tag{2.13}$$

$$\frac{\sum_{k\le n} a_{[b_k/c]} b_k}{b_{n+1}} \to 0, \tag{2.14}$$

where $[x]$ is the maximal integer not greater than $x \in \mathbb{R}$.

Proof: Without loss of generality, we may assume $c = 1$.

Let $c_1 = 1$ and for each $n \ge 1$,

$$c_{n+1} = c_n(1 + \sqrt{a_{[c_n]}}) = \prod_{k\le n}(1 + \sqrt{a_{[c_n]}}).$$

Then $c_n \uparrow \infty$. In fact, if $c_n \le M < \infty$ for all n, then

$$c_{n+1} = \prod_{k\le n}(1+\sqrt{a_{[c_n]}}) \ge \prod_{k\le n}(1 + \min_{1\le i\le M}\sqrt{a_i}) \ge (1 + \min_{1\le i\le M}\sqrt{a_i})^n \to \infty \quad \text{as } n\to\infty;$$

this yields a contradiction.

By Stolz's Theorem in analysis, we have

$$\lim_{n\to\infty}\frac{\sum_{k\le n} a_{[c_k]}c_k}{c_{n+1}} = \lim_{n\to\infty}\frac{\sum_{k\le n} a_{[c_k]}c_k - \sum_{k\le n-1} a_{[c_k]}c_k}{c_{n+1}-c_n} = \lim_{n\to\infty}\frac{a_{[c_n]}c_n}{c_{n+1}-c_n} = \lim_{n\to\infty}\sqrt{a_{[c_n]}} \to 0,$$

since $c_n \to \infty$ as $n \to \infty$.

Now, let $b_n = nc_n$; then

$$n/b_n = n/nc_n = 1/c_n \to 0 \quad \text{since } c_n \to \infty,$$

$$\lim_{n\to\infty} b_{n+1}/b_n = \lim_{n\to\infty} (n+1)/n \cdot \lim_{n\to\infty} c_{n+1}c_n = 1.$$

Since $a_n \downarrow 0$, and $b_k \geq c_k$,

$$0 < \frac{\sum_{k\leq n} a_{[b_k]} b_k}{b_{n+1}} \leq \frac{\sum_{k\leq n} \frac{k}{n+1} a_{[c_k]} c_k}{c_{n+1}} \leq \frac{\sum_{k\leq n} a_{[c_k]} c_k}{c_{n+1}} \to 0. \qquad \blacksquare$$

It follows from (2.2) that $(\rho')^n \mathcal{H}^s(E) \leq \mathcal{H}^s(E_{i_1\cdots i_n})$. Take $\rho_1 > 0$ small enough; then we have $(\rho_1)^n \leq \mathcal{H}^s(E_{i_1\cdots i_n})$, therefore

$$n \geq \frac{\log \mathcal{H}^s(E_{i_1\cdots i_n})}{\log \rho_1}. \tag{2.15}$$

For $c = \log \rho_1$, $a_n = \delta_n$ defined in Lemma 1, applying Lemma 2, we get a sequence $b_k \uparrow \infty$ satisfying

$$n/b_n \to 0, \quad b_{n+1}/b_n \to 1 \quad \text{and} \quad \Big(\sum_{k\leq n} a_{[b_k/c]} b_k/\Big) b_{n+1} \to 0. \tag{2.16}$$

Let $\varepsilon_k = e^{-b_k}$; then

$$\varepsilon_1 > \varepsilon_2 > \cdots > \varepsilon_n > \varepsilon_{n+1} > \cdots \to 0.$$

By (2.16), as $n \to \infty$,

$$\frac{n}{\log \varepsilon_n} \to 0, \quad \frac{\log \varepsilon_{n+1}}{\log \varepsilon_n} \to 1 \quad \text{and} \quad \frac{\sum_{k\leq n} \lambda_k \log \varepsilon_k}{\log \varepsilon_{n+1}} \to 0, \tag{2.17}$$

where

$$\lambda_k = \delta_{[\log \varepsilon_k / \log \rho_1]} \downarrow 0.$$

Let

$$\mathcal{A}_k = \{\mathbf{i} = i_1 \cdots i_n \in \Sigma^* : \mathcal{H}^s(E_{i_1\cdots i_n}) \leq \varepsilon_k \text{ and } \mathcal{H}^s(E_{i_1\cdots i_{n-1}}) > \varepsilon_k\}.$$

Then $\mathcal{A}_k$ is a cut-set for any $k \geq N = \min\{n : \mathcal{H}^s(E) > \varepsilon_n\}$.

Remark 5: Because of the lack of the $C^{1+\gamma}$ condition, for $\mathbf{i} \in \mathcal{A}_k$, maybe $\mathcal{H}^s(E_{\mathbf{i}}) \leq \varepsilon_{k'}$ for a large number k'. We can always find a largest number $k_1 \geq k$ such that $\mathbf{i} \in \mathcal{A}_{k_1}$ and $\mathbf{i} \notin \mathcal{A}_{(k_1+1)}$ since $\varepsilon_{n+1} < \varepsilon_n$ for all n.

Lemma 3: *If* $\mathbf{i} \in \mathcal{A}_k$, *then*

$$(\varepsilon_k)^{1+\lambda_k} \leq \mathcal{H}^s(E_{\mathbf{i}}) \leq \varepsilon_k.$$

Proof: For $\mathbf{i} = i_1 \cdots i_n \in \mathcal{A}_k$, $\mathcal{H}^s(E_{i_1 \cdots i_n}) \leq \varepsilon_k$. Then it follows from (2.15) that

$$n \geq \frac{\log \mathcal{H}^s(E_{i_1 \cdots i_n})}{\log \rho_1} \geq \frac{\log \varepsilon_k}{\log \rho_1}.$$

By Lemma 1,

$$\mathcal{H}^s(E_{i_1 \cdots i_n}) \geq \mathcal{H}^s(E_{i_1 \cdots i_{n-1}})^{1+\delta_n} \geq (\varepsilon_k)^{1+\delta_n},$$

where $n \geq [\frac{\log \varepsilon_k}{\log \rho_1}]$ and thus

$$\mathcal{H}^s(E_{i_1 \cdots i_n}) \geq (\varepsilon_k)^{1+\delta_n} \geq \varepsilon_k^{1+\lambda_k},$$

where $\lambda_k = \delta_{[\log \varepsilon_k / \log \rho_1]} \downarrow 0$. ∎

Lemma 1: *As* $|\mathbf{j}| \to \infty$,

$$\text{(2.18)} \qquad \frac{s \log |\varphi_{\mathbf{j}}(x_1) - \varphi_{\mathbf{j}}(x_2)|}{\log \mathcal{H}^s(E_{\mathbf{j}})} \to 1 \textit{ uniformly},$$

whenever $x_1 \in E_{i_1}$, $x_2 \in E_{i_2}$ *with* $1 \leq i_1 \neq i_2 \leq m$.

Proof: For $\mathbf{j} = \mathbf{j}'(j_1 \cdots j_{k_0})$, by using (2.10), we have

$$\min_{x \in \bar{V}} \|D_x \varphi_{\mathbf{j}'}\| (\rho')^{k_0} d(E_{i_1}, E_{i_2}) \leq |\varphi_{\mathbf{j}'(j_1 \cdots j_{k_0})}(x_1) - \varphi_{\mathbf{j}'(j_1 \cdots j_{k_0})}(x_2)|$$
$$\leq \max_{x \in \bar{V}} \|D_x \varphi_{\mathbf{j}'}\| (\rho)^{k_0} diam(E).$$

Therefore, (2.18) follows from (2.9), (2.11). ∎

3. Proof of Theorem 1

The proof is based on a bijection between the finite cut-sets of Σ^* and Π^*. For each $k \geq N = \min\{n : \mathcal{H}^s(E) > \varepsilon_n\}$, we will construct cut-sets $\mathfrak{F}_k$ of Σ^* and $\mathfrak{G}_k$ of Π^* **by induction**.

First step of induction for $k = N$: For $k = N$, considering the cut-set $\mathfrak{F}_N = \mathcal{A}_N$, we have

$$\text{(3.1)} \qquad \sum_{\mathbf{i} \in \mathfrak{F}_N} \mathcal{H}^s[E_{\mathbf{i}}] = \mathcal{H}^s[E].$$

By Lemma 3, for $\mathbf{i} \in \mathfrak{F}_N$,

$$\varepsilon_N^{1+\lambda_N} \leq \mathcal{H}^s[E_{\mathbf{i}}] \leq \varepsilon_N. \tag{3.2}$$

It follows from (3.1) and (3.2) that

$$\mathcal{H}^s[E]\varepsilon_N^{-1} \leq \#\mathfrak{F}_N = \sum_{\mathbf{i}\in\mathfrak{F}_N} 1 \leq \mathcal{H}^s[E]\varepsilon_N^{-(1+\lambda_N)}. \tag{3.3}$$

Write $|\mathfrak{F}_N| = 2^p + t$ with $p, t \in \mathbb{N} \cup \{0\}$ and $0 \leq t < 2^p$.

Then we can find a cut-set $\mathfrak{G}_N$ of Π^* with $|\mathfrak{G}_N| = |\mathfrak{F}_N|$ and $|\mathbf{j}| = p$ or $(p+1)$ for each $\mathbf{j} \in \mathfrak{G}_N$. Therefore,

$$2^{-(p+1)} \leq 2^{-|\mathbf{j}|} \leq 2^{-p} \quad \text{for each } \mathbf{j} \in \mathfrak{G}_N. \tag{3.4}$$

Since

$$(\#\mathfrak{F}_N)^{-1} \leq 2^{-p} \leq 2(\#\mathfrak{F}_N)^{-1}, \tag{3.5}$$

we have

$$\frac{1}{2}(\#\mathfrak{F}_N)^{-1} \leq 2^{-|\mathbf{j}|} \leq 2(\#\mathfrak{F}_N)^{-1}. \tag{3.6}$$

As a result, (3.3) gives

$$\varepsilon_N^{1+\lambda_N}/(2\mathcal{H}^s(E)) \leq 2^{-|\mathbf{j}|} \leq 2 \cdot \varepsilon_N/\mathcal{H}^s(E) \tag{3.7}$$

for each $\mathbf{j} \in \mathfrak{G}_N$.

Since $\#\mathfrak{F}_N = \#\mathfrak{G}_N$, we can get a one-to-one correspondence between $\mathfrak{F}_N$ and $\mathfrak{G}_N$. Let $h_N \colon \mathfrak{F}_N \to \mathfrak{G}_N$ denote the one-to-one mapping.

Induction for $k > N$: For $k > N$, suppose $\mathfrak{F}_{k-1}$ and $\mathfrak{G}_{k-1}$ have been constructed satisfying, for any $\mathbf{i} \in \mathfrak{F}_{k-1}$,

$$\mathbf{i} \in \mathcal{A}_r \text{ and } \mathbf{i} \notin \mathcal{A}_{r+1} \text{ with } r \geq k-1 \text{ dependent on } \mathbf{i}, \tag{3.8}$$

and there is a one-to-one mapping $h_{k-1} \colon \mathfrak{F}_{k-1} \to \mathfrak{G}_{k-1}$, which implies $\#\mathfrak{F}_{k-1} = \#\mathfrak{G}_{k-1}$.

For each $\mathbf{i} \in \mathfrak{F}_{k-1}$, suppose $\mathbf{i} \in \mathcal{A}_r$ and $\mathbf{i} \notin \mathcal{A}_{r+1}$, where $r \geq k-1$ is dependent on $\mathbf{i}$. Then the cylinder

$$[\mathbf{i}] = \bigcup_{\mathbf{i} \prec i', \mathbf{i}' \in \mathcal{A}_{r+1}} [\mathbf{i}'] \tag{3.9}$$

where $\mathbf{i}_1 \prec \mathbf{i}_2$ means $\mathbf{i}_1$ is a prefix of $\mathbf{i}_2$. Let

$$\mathfrak{F}_k = \{\mathbf{i}' : \mathbf{i} \prec \mathbf{i}' \text{ and } \mathbf{i}' \in \mathcal{A}_{r+1} \text{ for some } \mathbf{i} \in \mathfrak{F}_{k-1} \text{ with } \mathbf{i} \in \mathcal{A}_r \backslash \mathcal{A}_{r+1}\}.$$

Since the union in (3.9) is disjoint, we have

$$\mathcal{H}^s(E_{\mathbf{i}}) = \sum_{\mathbf{i} \prec \mathbf{i}', \mathbf{i}' \in \mathcal{A}_{r+1}} \mathcal{H}^s(E'_{\mathbf{i}}). \tag{3.10}$$

It follows from Lemma 3 that for $\mathbf{i} \in \mathcal{A}_r$, $\mathbf{i}' \in \mathcal{A}_{r+1}$ with $r \geq k-1$,

$$(\varepsilon_r)^{1+\lambda_r} \leq \mathcal{H}^s(E_{\mathbf{i}}) \leq \varepsilon_r, \quad (\varepsilon_{r+1})^{1+\lambda_r} \leq \mathcal{H}^s(E'_{\mathbf{i}}) \leq \varepsilon_{r+1}; \tag{3.11}$$

here $\lambda_r \geq \lambda_{r+1}$.

Applying the above estimations (3.11) to (3.10), we have

$$\frac{(\varepsilon_r)^{1+\lambda_r}}{\varepsilon_{r+1}} \leq \#\{\mathbf{i}' \in \mathcal{A}_{r+1} : \mathbf{i} \prec \mathbf{i}'\} \leq \frac{\varepsilon_r}{(\varepsilon_{r+1})^{1+\lambda_r}}. \tag{3.12}$$

Write $\#\{\mathbf{i}' \in \mathcal{A}_{r+1} : \mathbf{i} \prec i'\} = 2^q + l$ with $q, l \in \mathbb{N} \cup \{0\}$ and $0 \leq l < 2^q$. Then we can find a decomposition of the cylinder $[h_{k-1}(\mathbf{i})]$,

$$[h_{k-1}(\mathbf{i})] = \bigcup_{\mathbf{j}_k \in \Pi^*} [h_{k-1}(\mathbf{i})\mathbf{j}_k], \tag{3.13}$$

such that the number of cylinders on the right of (3.13) is $2^q + l$, and $|\mathbf{j}_k| = q$ or $q+1$ for each $\mathbf{j}_k$ in the above union. For this $\mathbf{i} \in \mathfrak{F}_{k-1}$, let $\Lambda_{\mathbf{i}}^k$ be the set of all the $\mathbf{j}_k$ in the union (3.13).

Therefore, for each $\mathbf{j}_k \in \Lambda_{\mathbf{i}}^k$,

$$2^{-(q+1)} \leq 2^{-|\mathbf{j}_k|} \leq 2^{-q}. \tag{3.14}$$

Since

$$(\#\{\mathbf{i}' \in \mathcal{A}_{r+1} : \mathbf{i} \prec i'\})^{-1} \leq 2^{-q} \leq 2(\#\{\mathbf{i}' \in \mathcal{A}_{r+1} : \mathbf{i} \prec \mathbf{i}'\})^{-1},$$

we have

$$\frac{(\varepsilon_{r+1})^{1+\lambda_r}}{2\varepsilon_r} \leq 2^{-|\mathbf{j}_k|} \leq 2\frac{\varepsilon_{r+1}}{(\varepsilon_r)^{1+\lambda_r}} \tag{3.15}$$

for each $\mathbf{j}_k \in \Lambda_{\mathbf{i}}^k$.

We provide the following decomposition:

$$C = \bigcup_{\mathbf{i} \in \mathfrak{F}_{k-1}} \bigcup_{\mathbf{j}_k \in \Lambda_{\mathbf{i}}^k} [h_{k-1}(\mathbf{i})\mathbf{j}_k]. \tag{3.16}$$

Let $\mathfrak{G}_k = \{h_{k-1}(\mathbf{i})\mathbf{j}_k\}_{\mathbf{i}\in\mathfrak{F}_{k-1},\mathbf{j}_k\in\Lambda_{\mathbf{i}}^k}$. Then $\mathfrak{G}_k$ is a cut-set of Π^*.

Since $\#\{\mathbf{i}' \in \mathcal{A}_{r+1} : \mathbf{i} \prec i'\} = \#\Lambda_{\mathbf{i}}^k$, then a one-to-one mapping h_k can be defined naturally satisfying

$$h_{k-1}(\mathbf{i}) \prec h_k(\mathbf{i}') \tag{3.17}$$

whenever $\mathbf{i} \in \mathfrak{F}_{k-1}, \mathbf{i}' \in \mathfrak{F}_k$ with $\mathbf{i} \prec \mathbf{i}'$.

Now, we can write every element of E in the following form:

$$x = \varphi_{\mathbf{i}_N\mathbf{i}_{N+1}\cdots\mathbf{i}_k\cdots}(E) \tag{3.18}$$

where $\mathbf{i}_N\mathbf{i}_{N+1}\cdots\mathbf{i}_k \in \mathfrak{F}_k$ for all $k \geq N$. At the same time, we can write each element of C in the following form:

$$y = \mathbf{j}_N\mathbf{j}_{N+1}\cdots\mathbf{j}_k\cdots \tag{3.19}$$

where $\mathbf{j}_N\mathbf{j}_{N+1}\cdots\mathbf{j}_k \in \mathfrak{G}_k$ for all $k \geq N$.

By (3.17), a bijection f from E to C can be defined by

$$f(x) = \lim_{k\to\infty}(h_k(\mathbf{i}_N\mathbf{i}_{N+1}\cdots\mathbf{i}_k)\cdots) \quad \text{for } x = \varphi_{\mathbf{i}_N\mathbf{i}_{N+1}\cdots\mathbf{i}_k\cdots}(E). \tag{3.20}$$

Now, we will verify (2.1): Suppose

$$x = \varphi_{\mathbf{i}_N\mathbf{i}_{N+1}\cdots\mathbf{i}_k\mathbf{i}_{k+1}\cdots}(E) \quad \text{and} \quad x' = \varphi_{\mathbf{i}_N\mathbf{i}_{k_1}\cdots\mathbf{i}_k\mathbf{i}'_{k+1}\cdots}(E), \tag{3.21}$$

where $\mathbf{i}_{k+1} \neq \mathbf{i}'_{k+1}$.

For $N \leq j \leq k$, assume $\mathbf{i}_N\mathbf{i}_{N+1}\cdots\mathbf{i}_j \in \mathcal{A}_{r_j}\backslash\mathcal{A}_{(r_j+1)}$; then

$$r_N < \cdots < r_{k-1} < r_k. \tag{3.22}$$

Write $\mathbf{i}_{k+1} = \mathbf{i}i_l\cdots$, $\mathbf{i}'_{k+1} = \mathbf{i}i'_l\cdots$ with $1 \leq i_l \neq i'_l \leq m$, where $\mathbf{i}$ is the common prefix of $\mathbf{i}_{k+1}$ and $\mathbf{i}'_{k+1}$.

By the process of construction,

$$\mathbf{i}_N\cdots\mathbf{i}_k \in \mathcal{A}_{r_k} \quad \text{and} \quad \mathbf{i}_N\cdots\mathbf{i}_k\mathbf{i}_{k+1} \in \mathcal{A}_{(r_k+1)}. \tag{3.23}$$

Since $\mathbf{i} \prec \mathbf{i}_{k+1}$ and $\lambda_{r_k} \geq \lambda_{(r_k+1)}$, it follows from Lemma 3 that

$$(\varepsilon_{r_k+1})^{1+\lambda_{r_k}} \leq \mathcal{H}^s(E_{\mathbf{i}_N\cdots\mathbf{i}_k\mathbf{i}_{k+1}}) \leq \mathcal{H}^s(E_{\mathbf{i}_N\cdots\mathbf{i}_k\mathbf{i}}) \leq \mathcal{H}^s(E_{\mathbf{i}_N\cdots\mathbf{i}_k}) \leq \varepsilon_{r_k},$$

which means

$$\frac{\log\mathcal{H}^s(E_{\mathbf{i}_N\cdots\mathbf{i}_k\mathbf{i}})}{\log\varepsilon_{r_k}} \to 1 \quad \text{as } k \to \infty, \tag{3.24}$$

since $r_k \geq k$, $\lim_{k\to\infty} \lambda_{r_k} = 0$ and $\lim_{k\to\infty} \log \varepsilon_{(r_k+1)}/\log \varepsilon_{r_k} = 1$ by (2.17).

It follows from Lemma 4 and (3.24) that

$$\frac{s \log |x - x'|}{\log \varepsilon_{r_k}} = \Big(\frac{\log \mathcal{H}^s(E_{\mathbf{i}_N\cdots\mathbf{i}_k\mathbf{i}})}{\log \varepsilon_{r_k}}\Big)\Big(\frac{s\log|x-x'|}{\log \mathcal{H}^s(E_{\mathbf{i}_N\cdots\mathbf{i}_k\mathbf{i}})}\Big) = 1 + o(1), \tag{3.25}$$

where $o(1) \to 0$ uniformly as $k \to \infty$; here, $r_k \geq k$ by induction.

We will estimate the distance $d(f(x), f(x'))$ as follows. Write

$$f(x) = \mathbf{j}_N\mathbf{j}_{N+1}\cdots\mathbf{j}_k\mathbf{j}_{k+1}\cdots \quad \text{and} \quad f(x') = \mathbf{j}_N\mathbf{j}_{N+1}\cdots\mathbf{j}_k\mathbf{j}'_{k+1}\cdots \tag{3.26}$$

with $\mathbf{j}_{k+1} \neq \mathbf{j}'_{k+1}$. Here

$$2^{-[|\mathbf{j}_N|+\cdots+|\mathbf{j}_k|+|\mathbf{j}_{k+1}|]} \leq d(f(x), f(x')) \leq 2^{-(|\mathbf{j}_N|+\cdots+|\mathbf{j}_k|)}. \tag{3.27}$$

From the process of construction, we notice that for $N < p \leq k$,

$$\mathbf{j}_N\mathbf{j}_{N+1}\cdots\mathbf{j}_p \in \mathcal{A}_{r_p} \quad \text{and} \quad \mathbf{j}_N\mathbf{j}_{N+1}\cdots\mathbf{j}_p \in \mathcal{A}_{(r_{p-1}+1)}.$$

By Lemma 3, since $\lambda_{r_p}, \lambda_{r_{(p-1)}+1} \leq \lambda_{r_{(p-1)}}$, we have

$$(\varepsilon_{r_p})^{1+\lambda_{r_{p-1}}} \leq \mathcal{H}^s(E_{\mathbf{j}_N\mathbf{j}_{N+1}\cdots\mathbf{j}_p}) \leq (\varepsilon_{r_p}),$$
$$(\varepsilon_{(r_{p-1}+1)})^{1+\lambda_{r_{p-1}}} \leq \mathcal{H}^s(E_{\mathbf{j}_N\mathbf{j}_{N+1}\cdots\mathbf{j}_p}) \leq (\varepsilon_{(r_{p-1}+1)}).$$

Therefore,

$$\varepsilon_{(r_{p-1}+1)} \leq (\varepsilon_{r_p})^{\frac{1}{1+\lambda_{r_{p-1}}}} \quad \text{and} \quad (\varepsilon_{(r_{p-1}+1)})^{1+\lambda_{r_{p-1}}} \geq (\varepsilon_{r_p})^{(1+\lambda_{r_{p-1}})^2}. \tag{3.28}$$

Applying (3.28) to (3.15), for $N < p < k+1$, we get

$$\frac{(\varepsilon_{r_p})^{(1+\lambda_{r_{p-1}})^2}}{2\varepsilon_{r_{p-1}}} \leq 2^{-|\mathbf{j}_p|} \leq 2\frac{(\varepsilon_{r_p})^{\frac{1}{1+\lambda_{r_{p-1}}}}}{(\varepsilon_{r_{p-1}})^{(1+\lambda_{r_{p-1}})}}. \tag{3.29}$$

Since $r_{p-1} \geq (p-1)$, we have $\lambda_{r_{p-1}} \leq \lambda_{p-1}$. Then by (3.28) and (2.17),

$$\frac{\log \varepsilon_{r_p}}{\log \varepsilon_{r_{p-1}}} = \frac{\log \varepsilon_{r_p}}{\log \varepsilon_{(r_{p-1}+1)}}\frac{\log \varepsilon_{(r_{p-1}+1)}}{\log \varepsilon_{r_{p-1}}} \to 1 \text{ uniformly} \quad \text{as } p \to \infty. \tag{3.30}$$

In particular, for any $p > N$,

$$\Big|\frac{\log \varepsilon_{r_p}}{\log \varepsilon_{r_{p-1}}}\Big| \leq C_0 \tag{3.31}$$

where C_0 is an independent constant.

For $p=k+1$, using (3.15), we have

$$\frac{(\varepsilon_{r_k+1})^{1+\lambda_{r_k}}}{2\varepsilon_{r_k}} \le 2^{-|\mathbf{j}_{k+1}|} \le 2\frac{\varepsilon_{r_k+1}}{(\varepsilon_{r_k})^{1+\lambda_{r_k}}}. \tag{3.32}$$

By the process of construction, there are constants $C_1, C_2 > 0$ so that

$$C_1 \le 2^{-|\mathbf{j}_N|}, \quad \varepsilon_{r_N} \le C_2. \tag{3.33}$$

It follows from (3.27), (3.29), (3.32) and (3.33) that

$$\begin{aligned} C_1\Big[\prod_{N<p\le k}\frac{(\varepsilon_{r_p})^{(1+\lambda_{r_{p-1}})^2}}{2\varepsilon_{r_{p-1}}}\Big]\frac{(\varepsilon_{r_k+1})^{1+\lambda_{r_k}}}{2\varepsilon_{r_k}} &\le d(f(x),f(x')) \\ &\le C_2\prod_{N<p\le k}\Big[2\frac{(\varepsilon_{r_p})^{\frac{1}{1+\lambda_{r_{p-1}}}}}{(\varepsilon_{r_{p-1}})^{(1+\lambda_{r_{p-1}})}}\Big]. \end{aligned}$$

Here for any $p>N$, there is a constant $C_3>0$ such that

$$(1+\lambda_{r_{p-1}})^2 \le 1+C_3\lambda_{r_{p-1}}, \quad \frac{1}{1+\lambda_{r_{p-1}}} \ge 1-C_3\lambda_{r_{p-1}}.$$

Therefore,

$$\begin{aligned} C_1\Big[\prod_{N<p\le k}\frac{(\varepsilon_{r_p})^{(1+C_3\lambda_{r_{p-1}})}}{2\varepsilon_{r_{p-1}}}\Big]\frac{(\varepsilon_{r_k+1})^{1+\lambda_{r_k}}}{2\varepsilon_{r_k}} &\le d(f(x),f(x')) \\ &\le C_2\prod_{N<p\le k}\Big[2\frac{(\varepsilon_{r_p})^{(1-C_3\lambda_{r_{p-1}})}}{(\varepsilon_{r_{p-1}})^{(1+\lambda_{r_{p-1}})}}\Big]. \end{aligned}$$

As a result,

$$\begin{aligned} &\Big[\frac{C_2}{\log\varepsilon_{r_k}} + \log 2\cdot\Big(\frac{k-N}{\log\varepsilon_{r_k}}\Big) + (1-C_3\lambda_{r_{k-1}}) \\ &\quad -(1+\lambda_{r_N})\frac{\log\varepsilon_{r_N}}{\log\varepsilon_{r_k}} - (C_3+1)\frac{\sum_{N<p<k}\lambda_{r_{p-1}}\log\varepsilon_{r_p}}{\log\varepsilon_{r_k}}\Big] \\ &\qquad \le \frac{\log d(f(x),f(x'))}{\log\varepsilon_{r_k}} \\ &\qquad \le \Big[\frac{C_1}{\log\varepsilon_{r_k}} - \log 2\cdot\Big(\frac{k-N+1}{\log\varepsilon_{r_k}}\Big) + (1+\lambda_{r_k})\frac{\log(\varepsilon_{r_k+1})}{\log\varepsilon_{r_k}} - \frac{\log\varepsilon_{r_N}}{\log\varepsilon_{r_k}} \\ &\qquad\quad + C_3\frac{\sum_{N<p\le k}\lambda_{r_{p-1}}\log\varepsilon_{r_p}}{\log\varepsilon_{r_k}}\Big]. \end{aligned}$$

Since $r_k \ge k$, by (2.17), then

$$\lim_{k\to\infty} \frac{k}{\log \varepsilon_{r_k}} = \lim_{k\to\infty} \frac{1}{\log \varepsilon_{r_k}} = 0, \quad \text{and} \quad \lim_{k\to\infty} \frac{\log(\varepsilon_{r_k+1})}{\log \varepsilon_{r_k}} = 1.$$

Moreover, using (2.17) and (3.31), we have

$$\begin{aligned}
\left|\frac{\sum_{N<p\le k} \lambda_{r_{p-1}} \log \varepsilon_{r_p}}{\log \varepsilon_{r_k}}\right| &= \left|\frac{\sum_{N<p\le k} \left[(\lambda_{r_{p-1}} \log \varepsilon_{r_{p-1}}) \cdot \frac{\log \varepsilon_{r_p}}{\log \varepsilon_{r_{p-1}}}\right]}{\log \varepsilon_{r_k}}\right| \\
&\le C_0 \left|\frac{\sum_{N<p\le k} (\lambda_{r_{p-1}} \log \varepsilon_{r_{p-1}})}{\log \varepsilon_{(r_k+1)}}\right| \cdot \left|\frac{\log \varepsilon_{(r_k+1)}}{\log \varepsilon_{r_k}}\right| \\
&\le C_0 \left|\frac{\sum_{1\le i<r_k} (\lambda_i \log \varepsilon_i)}{\log \varepsilon_{(r_k+1)}}\right| \cdot \left|\frac{\log \varepsilon_{(r_k+1)}}{\log \varepsilon_{r_k}}\right| \\
&\to 0 \quad \text{as } k \to \infty.
\end{aligned}$$

Consequently,

$$\frac{\log d(f(x), f(x'))}{\log \varepsilon_{r_k}} = 1 + o(1), \tag{3.34}$$

where $o(1) \to 0$ uniformly as $k \to \infty$ (since $r_k \ge k$).

Let $|x - x'| \to 0$; we have $k \to \infty$. Then (2.1) follows from (3.25) and (3.34).

4. Proof of Theorem 3

In this section, we will prove Theorem 3 which gives a necessary and sufficient condition for a self-similar arc to be a quasi-arc.

Proof of Theorem 3: Using Lemma 1 of [9], we may assume γ is a self-similar arc with two endpoints e_1, e_2, and there exists a family of contracting similitudes $\{S_i\}_{1\le i\le N}$ such that $\gamma = \bigcup_{i=1}^N S_i(\gamma)$,

$$e_1 = S_1(e_1), \quad e_2 = S_N(e_2)$$

and

$$S_i(\gamma) \cap S_j(\gamma) = \begin{cases} \varnothing & \text{if } |i-j| > 1, \\ \text{a singleton} & \text{if } |i-j| = 1. \end{cases}$$

Step 1: Suppose γ is a self-similar arc which is Hölder equivalent to the unit interval $[0, 1]$; we will show that γ is a quasi-arc.

In fact, by Definition 4 on the Hölder equivalence, there are constants τ, $\beta > 0$ and a bijection $f\colon \gamma \to [0, 1]$ such that for all $x_1, x_2 \in \gamma$,

$$\tau^{-1}|x_1 - x_2| \le |f(x_1) - f(x_2)|^\beta \le \tau|x_1 - x_2|. \tag{4.1}$$

Since f is a bijection, for any different points $x, y \in \gamma$,

$$f[\gamma(x,y)] = [f(x), f(y)] \text{ or } [f(y), f(x)]. \tag{4.2}$$

Therefore, using (4.1) and (4.2), we have

$$\begin{aligned} diam[\gamma(x,y)]] &\leq \sup_{x_1,x_2\in\gamma(x,y)} |x_1 - x_2| \leq \tau \sup_{x_1,x_2\in\gamma(x,y)} |f(x_1) - f(x_2)|^\beta \\ &\leq \tau |f(x) - f(y)|^\beta \\ &= \tau^2 |x - y|. \end{aligned}$$

That means γ is a quasi-arc. Then Step 1 is completed.

STEP 2: Suppose γ is a self-similar quasi-arc; we will show that γ is Hölder equivalent to the unit interval $[0, 1]$.

As γ is a quasi-arc, there is a constant $C > 0$ satisfying

$$diam(\gamma(x,y)) \leq C|x - y| \quad \text{for all } x, y \in \gamma. \tag{4.3}$$

Suppose $\gamma = \bigcup_{i=1}^N S_i(\gamma)$, where the ratio of S_i is ρ_i for all i and s is defined by the equality $\sum_{i=1}^N \rho_i^s = 1$. Write $w_i = \rho_i^s$. Let μ be the self-similar probability measure on γ such that

$$\mu = \sum_{i=1}^N w_i(\mu \circ S_i^{-1}). \tag{4.4}$$

Then

$$\mu(S_{i_1\cdots i_k}(\gamma)) = w_{i_1} \cdots w_{i_k}. \tag{4.5}$$

The mapping $f\colon \gamma \to [0, 1]$ is defined by

$$f(x) = \mu[\gamma(e_1, x)], \tag{4.6}$$

where e_1 is an endpoint of γ. Then

$$|f(x) - f(y)| = \mu[\gamma(x,y)]. \tag{4.7}$$

Since $\mu(\gamma') > 0$ for any nonempty subarc γ', $f(x) \neq f(y)$ for any different points $x, y \in \gamma$. Moreover, by (4.5) and the structure of a self-similar arc, $f\colon \gamma \to [0, 1]$ is surjective. Therefore, f is a bijection.

Given different points $x, y \in \gamma$, suppose there is a maximal sequence $i_1 \cdots i_k$ (maybe an empty sequence) such that $x, y \in S_{i_1\cdots i_k}(\gamma)$, but $x \in S_{i_1\cdots i_k i_{k+1}}(\gamma)$, $y \in S_{i_1\cdots i_k j_{k+1}}(\gamma)$ with $i_{k+1} \neq j_{k+1}$.

Without loss of generality, we assume $i_{k+1} < j_{k+1}$.

Case 1: $|i_{k+1} - j_{k+1}| > 1$.

In this case,

$$\min_{i_{k+1}<j<j_{k+1}} \mu[S_{i_1\cdots i_k j}(\gamma)] \le |f(x) - f(y)| = \mu[\gamma(x,y)] \le \mu[S_{i_1\cdots i_k}(\gamma)].$$

Using (4.5), we have

$$(\min_i w_i) \le \frac{|f(x) - f(y)|}{w_{i_1}\cdots w_{i_k}} \le 1. \tag{4.8}$$

On the other hand,

$$|x - y| = (\rho_{i_1}\cdots\rho_{i_k})|x' - y'|,$$

where $x' \in S_{i_{k+1}}(\gamma)$, $y' \in S_{j_{k+1}}(\gamma)$ satisfying

$$S_{i_1\cdots i_k}(x') = x, \quad S_{i_1\cdots i_k}(y') = y.$$

Thus,

$$\min_{|i-j|>1} d(S_i(\gamma), S_j(\gamma)) \le \frac{|x-y|}{\rho_{i_1}\cdots\rho_{i_k}} \le diam(\gamma). \tag{4.9}$$

Therefore, in this case,

$$\frac{\min_i w_i}{diam(\gamma)^s} \le \frac{|f(x) - f(y)|}{|x-y|^s} \le \Big[\min_{|i-j|>1} d(S_i(\gamma), S_j(\gamma))\Big]^{-s}. \tag{4.10}$$

Case 2: $|i_{k+1} - j_{k+1}| = 1$.

Suppose $b \in S_{i_1\cdots i_k i_{k+1}}(\gamma) \cap S_{i_1\cdots i_k j_{k+1}}(\gamma)$; then $b \in \gamma(x,y)$ and $b \neq x, b \neq y$.

Now, as $b \in \gamma(x,y)$,

$$|f(x) - f(y)| = |f(x) - f(b)| + |f(b) - f(y)|. \tag{4.11}$$

We will consider each term on the right of (4.11).

In fact, $b' = (S_{i_1\cdots i_k i_{k+1}})^{-1} b$ is an endpoint of γ, i.e., $b' \in \{e_1, e_2\}$. Without loss of generality, we assume $e_2 = b' = S_N(b')$, i.e., $\{b'\} = S_{NNN\cdots}(\gamma)$. (The proof is similar when $b' = e_1$.)

Suppose $x' = (S_{i_1\cdots i_k i_{k+1}})^{-1} x \in S_{j_1 j_2\cdots j_l\cdots}(\gamma)$ where j_l is the first symbol in $j_1 j_2\cdots$ not being N with $l \ge 1$, i.e., $j_1 j_2 \cdots j_l = \underbrace{NN\cdots N}_{l-1} j_l$.

Since $x, b \in S_{i_1\cdots i_k i_{k+1}}(\gamma)$ and μ is a self-similar measure by (4.4),

$$|f(x) - f(b)| = (w_{i_1}\cdots w_{i_k} w_{i_{k+1}})|f(x') - f(b')| \tag{4.12}$$

and

$$|x-b| = (\rho_{i_1}\cdots\rho_{i_k}\rho_{i_{k+1}})|x'-b'|. \tag{4.13}$$

Here

$$|f(x')-f(b')| \le \mu[S_{\underbrace{N\ldots N}_{l-1}}(\gamma)] \le (w_N)^{l-1} \tag{4.14}$$

and

$$|f(x')-f(b')| \ge \mu[S_{\underbrace{N\ldots N}_{l}}(\gamma)] \ge (w_N)^{l}. \tag{4.15}$$

On the other hand, since $e_2 = b'$,

$$(\rho_N)^{l-1}\min_{i\ne N} d(e_2, S_i(\gamma)) \le |x'-b'| \le (\rho_N)^{l-1} diam(\gamma). \tag{4.16}$$

From (4.12)–(4.16), we have

$$D_1^{-1}|x-b|^s \le |f(x)-f(b)| \le D_1|x-b|^s \tag{4.17}$$

for some constant $D_1 > 0$. Similarly, there is a constant $D_2 > 0$ so that

$$D_2^{-1}|b-y|^s \le |f(b)-f(y)| \le D_2|b-y|^s. \tag{4.18}$$

Let $D = \max(D_1, D_2)$; then by (4.11), (4.17) and (4.18),

$$D^{-1}(|x-b|^s + |b-y|^s) \le |f(x)-f(y)| \le D(|x-b|^s + |b-y|^s). \tag{4.19}$$

By (4.3), we have

$$C^{-1}(|x-b| + |b-y|) \le |x-y| \le |x-b| + |b-y|. \tag{4.20}$$

Since $s = \dim_H\gamma \ge 1$ (see [9]),

$$(|x-b|^s + |b-y|^s) \le (|x-b| + |b-y|)^s \le \kappa(|x-b|^s + |b-y|^s) \tag{4.21}$$

for some constant $\kappa > 0$ only dependent on s.

It follows from (4.19), (4.20) and (4.21) that in case 2,

$$M^{-1} \le \frac{|f(x)-f(y)|}{|x-y|^s} \le M \tag{4.22}$$

for some constant $M > 0$.

By (4.10) and (4.22), we have

$$\theta^{-1}|x-y| \le |f(x)-f(y)|^{1/s} \le \theta|x-y| \quad \text{for any } x, y \in \gamma,$$

where $\theta > 0$ is a constant. Then Step 2 is completed.

ACKNOWLEDGEMENT: The author is grateful to Professor Wen Zhi-Ying of Tsinghua University for his helpful comments.

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